Slender-Body Potential Flow between Closely Spaced Walls

Allen Plotkin*
University of Maryland, College Park, Md.

The problem under consideration is the unsteady potential flow of a slender body translating parallel to its long axis between closely spaced (separation of the order of the slenderness parameter) walls. The objective is to obtain analytically the unsteady contribution to the flowfield and pressure distribution due to streamwise variation in the wall separation. The solution also applies to a slender ship moving in variable-depth shallow water at zero Froude number. The method of matched asymptotic expansions is used to take advantage of the simplified governing equations in the near and far fields. The solution is given to second order for arbitrarily small deviation from a mean separation.

I. Introduction

THE presence of closely spaced boundaries can alter significantly the flowfield generated by a slender body moving in an ideal fluid. As an example, consider the potential flow of a slender body, symmetrically placed between parallel plane walls, translating parallel to its long axis. Tuck[1] analyzed this flow using the method of matched asymptotic expansions. (A slender ship translating in shallow water was considered and, mathematically, the situation is identical to the foregoing one for zero Froude number.) It was shown that the pressure acting on the body is an order of magnitude larger in the presence of the walls, and the flow far from the body is constrained to be two dimensional. Significant departure from the infinite-fluid solution also was found by Newman[2] in his treatment of the lateral flow past a slender body between parallel plane walls.

The flow is steady in the problems of Tuck and Newman when body-fixed coordinates are used. Plotkin[3,4] has extended the analysis of Tuck[1] to include the unsteady flow generated by a slender ship translating over a wavy bottom in shallow water. These studies have as their aerodynamic counterpart the translation of a slender body symmetrically placed between wavy walls whose amplitude is small compared to wall separation. In Ref. 3, the wall wavelength is of the order of the body length. In Ref. 4, the wavelength of the wall is taken to be $O(\epsilon^{1/2})$, where ϵ is the slenderness parameter, and the presence of dual characteristic lengths in the direction of motion led to the use of the method of multiple scales.[5] Plotkin[6] also has considered the unsteady shallow-water flow of a slender ship translating past an arbitrarily slowly varying bottom.

In this paper, the solution techniques developed by Tuck and Plotkin are used to obtain analytically the unsteady contribution to the potential flow generated by a slender body translating between closely spaced walls whose separation varies in the direction of motion. The variable component of the wall separation has small amplitude, a longitudinal length scale that is comparable to the body length, and is otherwise arbitrary. This work is also an extension to arbitrary bottom topography of the shallow-water solution of Plotkin[3] for the case of zero Froude number.

II. Problem Formulation

Consider a slender body of length 2ℓ translating parallel to its long axis with constant speed U as shown in Fig. 1. The body is symmetrically located between two closely spaced walls whose separation varies only in the direction of motion. A body-fixed Cartesian coordinate system is introduced with its origin at the center of the body axis; and the xy plane, with x opposite to the direction of motion, is taken as a plane of symmetry.

The body surface is given by $y=\epsilon f(x,z)$ in the moving system where ϵ is a measure of the ratio of the transverse body dimensions to the body length 2ℓ which is O(1). The speed U also is taken to be $O(1)$. Gauge functions are displayed explicitly so that all the variables in the paper are $O(1)$ unless otherwise stated. The walls are described by $z=\pm\epsilon h(x-Ut)$. (Since the walls are stationary in a space-fixed frame, their description is a function of $x-Ut$ in the moving frame). Since $z=0$ is a plane of flow symmetry, it is sufficient to study the flow in the region $0\leq z\leq \epsilon h$.

For incompressible irrotational flow the velocity is represented as the positive gradient of a velocity potential $\phi(x,y,z,t)$ which satisfies Laplace's equation. Kinematic boundary conditions must be satisfied on the wall and body and a symmetry condition applies at $z=0$. The complete system of equations is

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \qquad \text{in fluid domain} \tag{1}$$

$$\epsilon U f_x+\epsilon\phi_x f_x+\epsilon\phi_z f_z-\phi_y=0 \quad \text{on } y=\epsilon f \tag{2}$$

$$\epsilon\phi_x h_x-\phi_z=0 \qquad \text{on } z=\epsilon h \tag{3}$$

$$\phi_z=0 \qquad \text{on } z=0 \tag{4}$$

Since ϵ is an appropriate scale in the z direction, the coordinate $Z=z/\epsilon$ will be used in the analysis. The method of matched asymptotic expansions will be used to study the flow in regions near to, and far from, the body.

III. Analysis

The wall separation is

$$h=h_0+\epsilon h_1(x-Ut) \tag{5}$$

where h_0 is a constant and the length scale associated with the variation of h_1 is comparable to the body length.

A. Outer Expansion

The outer region, far from the body, is defined by the following order of magnitude of the coordinates with respect to body length

$$x,y=O(1), \qquad z=O(\epsilon) \tag{6}$$

The velocity potential is expanded in an asymptotic series in ϵ of the form

$$\phi=\epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\epsilon^3\phi^{(3)}+\epsilon^4\phi^{(4)}+\ldots \tag{7}$$

Received April 28, 1976; revision received Nov. 29, 1976.
Index categories: Hydrodynamics; Nonsteady Aerodynamics.

*Associate Professor of Aerospace Engineering. Member AIAA.

Substitution into Laplace's equation [Eq. (1)] yields

$$\phi_{ZZ}^{(1)}=\phi_{ZZ}^{(2)}=0, \quad \phi_{ZZ}^{(3)}=-\nabla^2\phi^{(1)}, \quad \phi_{ZZ}^{(4)}=-\nabla^2\phi^{(2)} \tag{8}$$

where the two-dimensional Laplacian is

$$\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2) \tag{9}$$

The wall and symmetry plane boundary conditions [Eqs. (3) and (4)] become

$$\phi_Z^{(1)}=\phi_Z^{(2)}=\phi_Z^{(3)}=0, \quad \phi_Z^{(4)}=h_{I_x}\phi_x^{(1)} \quad \text{on } Z=h \tag{10}$$

and

$$\phi_Z^{(1)}=\phi_Z^{(2)}=\phi_Z^{(3)}=\phi_Z^{(4)}=0 \quad \text{on} \quad Z=0 \tag{11}$$

Eq. (8) is integrated once with respect to Z and with the use of Eqs. (10) and (11) we get

$$\phi^{(1)}=\phi^{(1)}(x,y,t), \quad \phi^{(2)}=\phi^{(2)}(x,y,t) \tag{12a}$$

$$\nabla^2\phi^{(1)}=0, \quad \nabla^2\phi^{(2)}=-h_{I_x}\phi_x^{(1)}/h_0 \tag{12b}$$

Eq. (12 is solved formally by Green's function (source) distributions to yield

$$\phi^{(1)}=\int_{-\infty}^{\infty}\mu^{(1)}(\xi,t)\ \ G(x-\xi,y)\mathrm{d}\xi \tag{13}$$

and

$$\phi^{(2)}=\int_{-\infty}^{\infty}\mu^{(2)}(\xi,t)\ \ G(x-\xi,y)\mathrm{d}\xi$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{I_x}(\xi-Ut)\phi_x^{(1)}$$

$$\times(\xi,\eta,t)h_0^{-1}G(x-\xi,y-\eta)\mathrm{d}\xi\mathrm{d}\eta \tag{14}$$

where

$$G(x,y)=(2\pi)^{-1}\log(x^2+y^2)^{1/2} \tag{15}$$

and the source strengths $\mu^{(2)}\ (x,\ t)$ and $\mu^{(2)}\ (x,\ t)$ are as yet unknown.

It is noted that the differential equations for $\phi^{(1)}$ and $\phi^{(2)}$ contain the two-dimensional Laplace operator. Time essentially appears as a parameter so that no difficulty is involved in considering an arbitrary description for h_I. In the corresponding shallow-water problems in Plotkin[3,4] the free-surface boundary condition leads to differential equations containing the two-dimensional wave operator which includes time derivatives. This led to a consideration of problems that were harmonic in their time dependence.

B. Inner Expansion

The inner region, near the body, is defined by the following orders of magnitude of the coordinates with respect to the body length

$$x=O(1), \quad y,z=O(\epsilon) \tag{16}$$

The velocity potential is expanded in an asymptotic series in ϵ of the form

$$\phi=\epsilon\Phi^{(1)}+\epsilon^2\Phi^{(2)}+\epsilon^3\Phi^{(3)}+\ \ldots \tag{17}$$

Introduction of the inner variable $Y=y/\epsilon$ and substitution into Laplace's equation yields

$$\Phi_{YY}^{(1)}+\Phi_{ZZ}^{(1)}=\Phi_{YY}^{(2)}+\Phi_{ZZ}^{(2)}=0, \quad \Phi_{YY}^{(3)}+\Phi_{ZZ}^{(3)}=-\Phi_{XX}^{(1)} \tag{18}$$

On $Z=0$, we get

$$\Phi_Z^{(1)}=\Phi_Z^{(2)}=\Phi_Z^{(3)}=0 \tag{19}$$

On the body, $Y=f$, conventional slender-body theory gives the condition

$$\Phi_N^{(1)}=0, \quad \Phi_N^{(2)}=Uf_x(1+f_Z^2)^{-1/2},$$
$$\Phi_N^{(3)}=\Phi_x^{(1)}f_x(1+f_Z^2)^{-1/2} \tag{20}$$

where N is the normal in the crossflow plane expressed in inner variables. The wall condition, Eq. (3), is satisfied and the resulting equations are expanded about the mean wall position, $Z=h_0$, to give

$$\Phi_Z^{(1)}=\Phi_Z^{(2)}=0, \quad \Phi_Z^{(3)}=-h_I\Phi_{ZZ}^{(2)}(h_0) \tag{21}$$

$\Phi^{(1)}$ satisfies Laplace's equation in the crossflow plane with zero normal derivatives on all boundaries. Therefore

$$\Phi^{(1)}=\Phi^{(1)}(x,t) \tag{22}$$

and is arbitrary. $\Phi^{(2)}$ can be written as

$$\Phi^{(2)}=\Phi^{(21)}(x,t)+\Phi^{(22)}(Y,Z;x) \tag{23}$$

where $\Phi^{(21)}$ is arbitrary. $\Phi^{(22)}$ satisfies a two-dimensional Neumann problem with a nonzero boundary condition on the body. A suitable boundary condition at infinity is

$$\Phi^{(22)}\rightarrow u(x)\ |Y|+o(1) \quad \text{as} \quad |Y|\rightarrow\infty \tag{24}$$

where u is determinable from conservation of mass (the flux from the body is $US'(x)$) as

$$u=US'(x)/4h_0 \tag{25}$$

and $\epsilon^2S(x)$ is the body cross-sectional area. $\Phi^{(3)}$ can be written as

$$\Phi^{(3)}=\Phi^{(31)}(x,t)+\Phi_x^{(1)}\Phi^{(22)}$$
$$+\Phi^{(33)}(Y,Z;x,t)-Y^2\Phi_{xx}^{(1)}/2+\phi^{(32)}(y,Z;x,t) \tag{26}$$

where $\Phi^{(31)}$ is arbitrary. The second term satisfies the homogeneous differential equation and homogeneous boundary conditions on $Z=0$ and h_0 and the body boundary condition for $\Phi^{(3)}$. $\Phi^{(33)}$ satisfies the homogeneous differential equation and homogeneous boundary conditions on $Z=0$ and $Y=f$ with a nonzero wall condition from Eq. (21). The wall flux is

$$\int_{-\infty}^{\infty}\Phi_Z^{(33)}\mathrm{d}Y=\int_{-\infty}^{\infty}h_I\Phi_{ZZ}^{(2)}\mathrm{d}Y=2uh_I$$

and therefore, for $|Y|\rightarrow\infty$

$$\Phi^{(33)}\rightarrow v(x,t)\ |Y|+o(1) \tag{27}$$

with

$$v=uh_I/h_0 \tag{28}$$

The fourth term satisfies the Poisson equation [Eq. (18)], homogeneous boundary conditions on $Z=0$ and h_0, and also contributes $-Y\Phi_{XX}^{(1)}(1+f_Z^2)^{-1/2}$ to the normal derivative on the body. $\Phi^{(32)}$ is introduced to correct this and satisfies the homogeneous differential equation and homogeneous boundary conditions on $Z=0$ and h_0, and

$$\Phi_N^{(32)}=Y\Phi_{xx}^{(1)}(1+f_Z^2)^{-1/2} \quad \text{on} \quad Y=f \tag{29}$$

This introduces the flux $\Phi_{xx}^{(1)}S(x)/2$ from the body and leads to

$$\Phi^{(32)}\to w(x,t)\,|Y|+o(1)\quad\text{as}\quad|Y|\to\infty \tag{30}$$

with

$$w=\Phi_{xx}^{(1)}S(x)/4h_0 \tag{31}$$

(It is noted that the constant-separation solution of Tuck[1] is in error since $\Phi^{(32)}$ is not included in the expression for $\Phi^{(1)}$. The expression for force is not affected.)

Matching the three-term inner expansion of the two-term outer expansion with the two-term outer expansion of the three-term inner expansion (see Van Dyke[7]) yields

$$\Phi^{(1)}=\Phi^{(1)}(y=0) \tag{32a}$$

$$\Phi^{(21)}=\phi^{(2)}(y=0) \tag{32b}$$

$$\mu^{(1)}=2u \tag{32c}$$

$$\mu^{(2)}=2[u\Phi_x^{(1)}+v+w] \tag{32d}$$

The velocity potential in the inner region, to $O(\epsilon^2)$, can then be written

$$\phi=\left\{\epsilon\phi^{(1)}(x,0)+\epsilon^2\left[(4\pi h_0)^{-1}\int_{-\infty}^{\infty}[US'(\xi)\phi_x^{(1)}(\xi,0)+S(\xi)\phi_{xx}^{(1)}(\xi,0)]\log|x-\xi|\,d\xi+\Phi^{(22)}(Y,Z;x)\right]\right\}$$
$$+\epsilon^2\left[U(4\pi h_0^2)^{-1}\int_{-\infty}^{\infty}h_1(\xi-Ut)S'(\xi)\log|x-\xi|\,d\xi-(2\pi h_0)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{1_x}(\xi-Ut)\phi_x^{(1)}(\xi,\eta)\times\log[(x-\xi)^2+\eta^2]^{1/2}\,d\xi\,d\eta\right] \tag{33}$$

where

$$\phi^{(1)}(x,y)=U(4\pi h_0)^{-1}\int_{-\infty}^{\infty}S'(\alpha)\log[(x-\alpha)^2+y^2]^{1/2}\,d\alpha \tag{34}$$

The first term, in braces, is the inner expansion of the constant-separation $(h=h_0)$ solution and will be denoted by $\Phi^{(0)}(x,Y,Z)$.

C. Pressure Distribution in Inner Region

The pressure is given by the Bernoulli equation as

$$p=-\rho[\phi_t+U\phi_x+(\phi_x^2+\phi_y^2+\phi_z^2)/2] \tag{35}$$

Using Eq. (33) the pressure is seen to be

$$p=p^{(0)}-\rho[(\partial/\partial t)+U(\partial/\partial x)](\phi-\Phi^{(0)}) \tag{36}$$

where $p^{(0)}$ is the pressure distribution, to $O(\epsilon^2)$, for constant separation $(h=h_0)$, and the second term, of $O(\epsilon^2)$, is the unsteady contribution due to the separation variation.

IV. Discussion and Results

The lowest-order unsteady contribution to the velocity potential due to rather general streamwise variation in wall separation has been obtained analytically. The results appear in Eqs. (33 and 34). To this order the velocity potential and therefore the pressure due to this contribution are functions only of the streamwise coordinate x and time t. The geometry of the body appears only through the cross-sectional area $S(x)$. The magnitude of the unsteady part of the velocity potential is $O(\epsilon^2)$.

The complete solution to the problem is the sum of the unsteady contribution and the constant-separation solution $\Phi^{(0)}$. $\Phi^{(0)}$ consists of two parts: an interaction part which is a function of x only and in which the cross-sectional area appears inside of the Green's function integrals, and a local part, $\Phi^{(22)}$, which varies in the crossflow plane.

The unsteady contribution to the flowfield depends on the streamwise derivative of the first-order constant-separation velocity potential

$$\epsilon\phi_x^{(1)}(x,0)=U\epsilon(4\pi h_0)^{-1}\int_{-\infty}^{\infty}\frac{S'(\alpha)\,d\alpha}{x-\alpha} \tag{37}$$

Also, the pressure, to $O(\epsilon)$, is

$$p=-\rho U\epsilon\phi_x^{(1)}(x,0) \tag{38}$$

Consider a symmetric body of revolution with a parabolic distribution in the symmetry plane.[3,4] The body description is

$$Y(Z=0)=f(x,0)=Y_0(1-x^2/\ell^2) \tag{39}$$

and

$$S(x)=\pi Y_0^2(1-x^2/\ell^2)^2$$

The first-order pressure is therefore

$$p=\rho U^2Y_0^2\epsilon(2\ell h_0)^{-1}\left\{4\frac{x^2}{\ell^2}-\frac{8}{3}+\frac{x}{\ell}\left(1-\frac{x^2}{\ell^2}\right)\times\log\left(\frac{x/\ell+1}{x/\ell-1}\right)^2\right] \tag{40}$$

Consider a wall with a symmetric parabolic bump with length equal to the body length. At $t=0$, the center of the bump is located at $x=0$. The wall separation is

$$h=h_0\qquad\left|\frac{x-Ut}{\ell}\right|>1$$
$$h=h_0\left[1-\epsilon\left(\frac{x-Ut}{\ell}\right)^2\right]\qquad\left|\frac{x-Ut}{\ell}\right|\le 1 \tag{41}$$

The unsteady pressure distribution is calculated numerically from Eqs. (33) and (36) and is plotted in Fig. 2 for values of Ut/ℓ of 0, 0.5 and 1. Let the pressure in Eq. (36) be

$$p=p^{(0)}+\epsilon^2p^{(1)}(x,t) \tag{42}$$

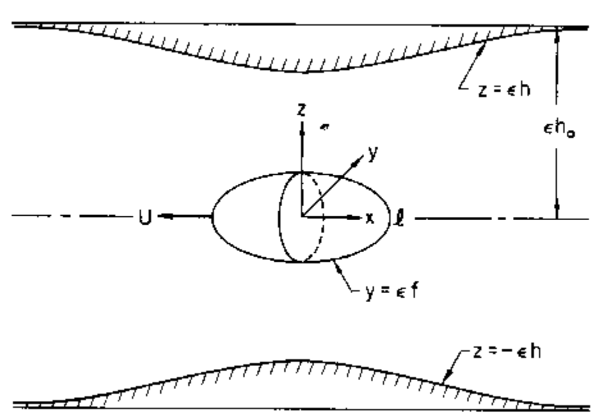

Fig. 1 Coordinate system.

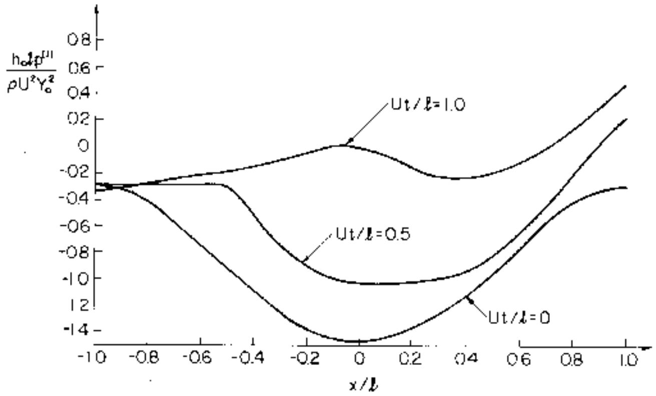

Fig. 2 Unsteady pressure distribution on a slender body translating between walls containing a symmetric parabolic bump.

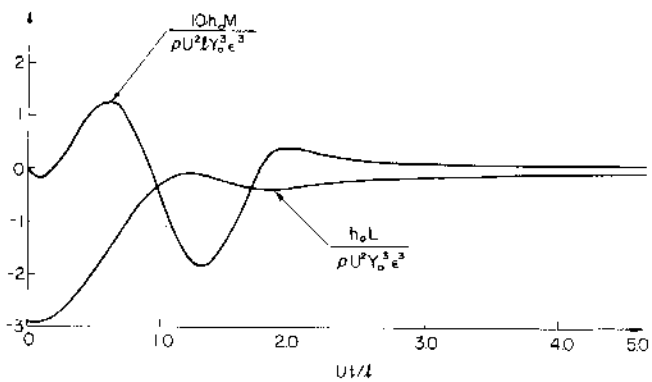

Fig. 3 Unsteady lift and moment acting on a ship with parabolic waterline passing over a parabolic bump in shallow water.

Then for the symmetric body and bump in the example

$$p^{(l)}(x,t) = p^{(l)}(-x,-t) \tag{43}$$

where $t<0$ when the center of the body axis is upstream of the center of the bump.

For the flow situation studied in this analysis, with the body placed symmetrically between the walls, the only unsteady force or moment which is nonzero is the drag force R. From Tuck[1], the unsteady drag force is given by

$$R = 2\epsilon^4 \int_{-\ell}^{\ell} p^{(l)}(x,t)\ S'(x)\,\mathrm{d}x \tag{44}$$

The solution obtained here also represents the variable-depth shallow-water flow of a slender ship at zero Froude number. $z=0$ is the rigid free surface and the body surface, for $z<0$ is the wetted portion of the hull. An unsteady lifting force L and pitching moment M then act on the ship in addition to the wave drag $R/2$ where R is given in Eq. (44). From Tuck[1]

$$L = 2\epsilon^3 \int_{-\ell}^{\ell} p^{(l)}(x,t)\quad f(x,0)\,\mathrm{d}x \tag{45a}$$

and

$$M = -2\epsilon^3 \int_{-\ell}^{\ell} xp^{(l)}(x,t)\ f(x,0)\,\mathrm{d}x \tag{45b}$$

where L is positive in the positive z direction and M is positive clockwise.

For a slender ship with the geometry of Eq. (39) passing over a parabolic bump with the geometry of Eq. (41), the lift and pitching moment are plotted vs time in Fig. 3. For the symmetry in the example, it is noted that L and M are symmetric and antisymmetric in time, respectively. For this example, the wave drag is related to the moment by

$$R/2 = \epsilon\pi Y_0 M/\ell^2 \tag{46}$$

The unsteady vertical force acts downwards for all time in this example and has a maximum value when the ship and bump are in a symmetrical configuration. The unsteady moment and wave drag are seen to change sign a number of times throuughout the course of the motion.

Acknowledgments

This research was performed while the author was a Visiting Associate in the Engineering Science Department of Caltech. It was supported by the National Science Foundation under Grant No. ENG 74-20573.

References

[1]Tuck, E.O., "Shallow-Water Flows Past Slender Bodies," *Journal of Fluid Mechanics*, Vol. 26, Sept. 1966, pp. 81-95.

[2]Newman, J.N., "Lateral Motion of a Slender Body Between Two Parallel Walls," *Journal of Fluid Mechanics*, Vol. 39, Oct. 1969, pp. 97-115.

[3]Plotkin, A., "The Flow Due to a Slender Ship Moving Over a Wavy Wall in Shallow Water," *Journal of Engineering Mathematics*, Vol. 10, July 1976, pp. 207-218.

[4]Plotkin, A., "Hydrodynamic Pressure Field on a Slender Ship Moving Over a High-Frequency Wavy Shallow Bottom," *Journal of Applied Mechanics*, Vol. 43, June 1976, pp. 232-236.

[5]Nayfeh, A.H., *Perturbation Methods*, Wiley, New York, 1973, Ch. 6

[6]Plotkin, A., "Slender Ship Shallow-Water Flow Past a Slowly Varying Bottom," *Journal of Engineering Mathematics*, (in press).

[7]Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964, p. 90.